TOPICS FROM THE GAUSSIAN WORLD

19 MAR 2020

CALENDARIO : GIO 19 MAR (OGGI)

MER 25 MAR

GIO 2 APR

MER 8 APR

GIO 16 APR

GIO 23 APR

HELLO, WORLD!

- PROCESSI STOCASTICI GAUSSIANI
  - TEORIA GENERALE VASTA, ELEGANTE, POTENTE
  - ESEMPI E MODELLI CONCRETI E SPECIFICI

- 2 PARTI (INDIPENDENTI, MA COLLEGATE)
  - → I (MAURIZIA) : APPROSSIMAZIONE GAUSSIANA (CLT)
  - → II (F.C.) : PROCESSI GAUSSIANI IN $\mathbb{R}^d$

PRESENTAZIONE DELLA II PARTE DEL CORSO (F.C.) ~ 5 LEZ.

- MOTO BROWNIANO $B = (B_t)_{t \in [0,1]}$

LEZ. 1 – RICHIAMI

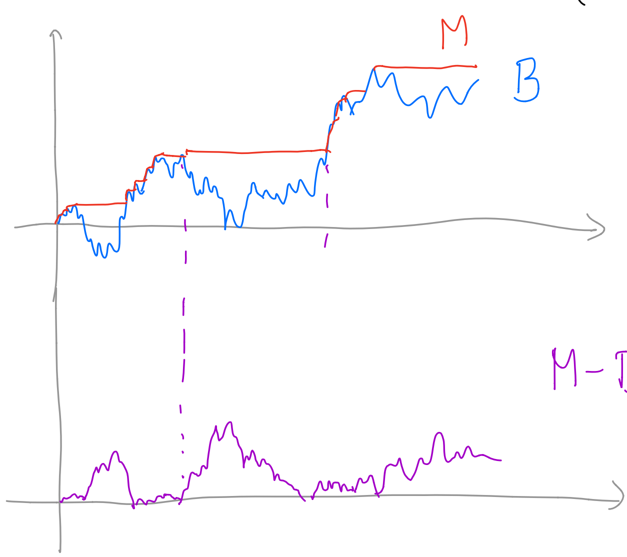
– PRINCIPIO DI RIFLESSIONE $\to$ M = MASSIMO DI B

$$M = \left(M_t := \max_{0 \le s \le t} B_s \right)_{t \in [0,1]}$$

$\forall$ t FISSATO, $M_t \sim |B_t|$

– TEOR. (LÉVY): $M - B \sim |B|$

$$M - B = \left(M_t - B_t \right)_{t \in [0,1]} \qquad |B| = \left(|B_t| \right)_{t \in [0,1]}$$



$M - B \overset{d}{=} |B|$

LEZ. 2 – PROPRIETÀ FRATTALI

$\to$ DIM. DI HAUSDORFF DI $A \subseteq \mathbb{R}^d$

– TEOR (TAYLOR): $\dim [\text{Zero}] = \frac{1}{2}$ Q.C. $(d=1)$

$\text{Zero} = \{ t \in [0,1] : B_t = 0 \}$

– TEOR (TAYLOR): B = M.B. IN $\mathbb{R}^d$, $d \geq 2$

$\text{Range} = \{ B_t : t \in [0,1] \} \subseteq \mathbb{R}^d$ $\qquad \dim [\text{Range}] = 2$ Q.C.

• GAUSSIAN FREE FIELD (GFF) – CAMPO LIBERO GAUSSIANO

$B = (B_t)_{t \in [0,\infty)}$ M.B. in $\mathbb{R}$

FISSO $N \in \mathbb{N}$.

$$\frac{P\left((B_1, B_2, \dots, B_N) \in (dx_1, \dots, dx_N)\right)}{dx_1 \cdots dx_N} \propto e^{-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{2}}$$

$$= e^{-\frac{1}{4} \sum_{\substack{i,j \in \{0,1,\dots,N\} \\ i \sim j}} (x_i - x_j)^2}$$

$i \sim j \iff |i-j| = 1$

$I = \{0, 1, \dots, N\}$

LEZ. 3

- GFF "DISCRETO" : DATO $(I, \sim)$ GRAFO FINITO

  ↳ PROC. GAUSSIANO $(X_i)_{i \in I}$

  $$\frac{P\left((X_i)_{i \in I} \in dx\right)}{dx} \propto \boxed{e^{-\frac{1}{4} \sum_{\substack{i,j \in I \\ i \sim j}} (x_i - x_j)^2}}$$

- PASSEGGIATE ALEATORIE
- FUNZIONI ARMONICHE

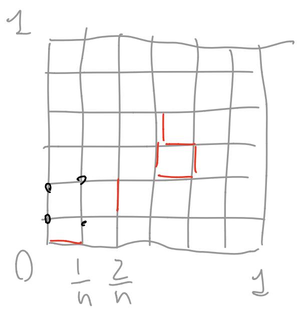
LEZ. 4 • GFF "CONTINUO" SU UN DOMINIO $D \subseteq \mathbb{R}^2$

$$D = [0,1] \times [0,1]$$

$\forall n \in \mathbb{N}$: $I_n := D \cap \frac{1}{n}\mathbb{Z}^2 = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\} \times \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$

$\Downarrow$

$z \sim z' \iff |z - z'| = \frac{1}{n}$



$X^{(n)} \sim$ GFF su I_n $\underset{n \to \infty}{\Longrightarrow}$ $X :=$ GFF su D

$\shortparallel$

$(X_z^{(n)})_{z \in I_n}$

$\Downarrow$

"DISTRIBUZIONE"

$\shortparallel$

FUNZIONE GENERALIZZATA SU $\mathbb{R}^2$

- LEZ. 5 WHITE NOISE = RUMORE BIANCO

  E/O GAUSSIAN MULTIPLICATIVE CHAOS $\approx e^X$

  E/O ABSTRACT WIENER SPACE

- RICHIAMI SU V.A. GAUSSIANE (O NORMALI)

$(\Omega, \mathcal{A}, P)$

- V.A. NORMALI IN $\mathbb{R}$

$\to$ UNA V.A. REALE Z SI DICE NORMALE STANDARD, $Z \sim N(0,1)$

$$\Leftrightarrow \qquad f_Z(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

→ UNA V.A. $X \sim N(m, \sigma^2) \Leftrightarrow X \sim m + \sigma Z$ CON $Z \sim N(0,1)$

▷ SE $\sigma^2 = 0$ ALLORA $X = m$ Q.C.

$$\Leftrightarrow \quad N(m, 0) = \delta_m \quad \text{(DIRAC)}$$

▷ SE $\sigma^2 > 0$ ALLORA X HA DENSITÀ

$$f_X(x) = \frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

→ FUNZIONE CARATTERISTICA $X \sim N(m, \sigma^2)$

$$E[e^{i\vartheta X}] = e^{i\vartheta m - \frac{1}{2}\vartheta^2\sigma^2}$$

→ X NORMALE $\Rightarrow aX + b$ NORMALE $\forall a, b \in \mathbb{R}$

→ TEOR. X E Y NORMALI INDIPENDENTI $\Rightarrow X+Y$ È NORMALE

- V.A. NORMALI IN $\mathbb{R}^d$

→ UN V.A. $X = (X_1, \dots, X_d) \in \mathbb{R}^d$ SI DICE NORMALE $\Leftrightarrow$

$\langle a, X \rangle = a_1 X_1 + \dots + a_d X_d$ È NORMALE IN $\mathbb{R}$, $\forall a \in \mathbb{R}^d$

→ SCRIVIAMO $X \sim N(m, K)$

m = VETTORE MEDIA $= E[X] = \left(m_i = E[X_i]\right)_{i=1,\dots,d}$

K = MATRICE DELLE COV. $= Cov[X \otimes X] = \left(K_{ij} := Cov[X_i, X_j]\right)_{1 \le i,j \le d}$

$m \in \mathbb{R}^d$

$K \in \mathbb{R}^{d \times d}$

→ FUNZ. CARATT. $X \sim N(m, K)$, ALLORA $\forall \vartheta \in \mathbb{R}^d$

$$E\left[e^{i\langle \vartheta, X\rangle}\right] = e^{i\langle \vartheta, m\rangle - \frac{1}{2}\langle \vartheta, K\vartheta\rangle}$$

→ X NORMALE $\Rightarrow$ $AX + b$ NORMALE $\forall A$ MATRICE, b VETTORI

$$X \sim N(m, K) \Rightarrow Y := AX + b \sim N(Am + b, AKA^*)$$

$$A^* = A^T$$

→ TEOR. X NORMALE IN $\mathbb{R}^d$, $Cov[X_i, X_j] = 0 \Rightarrow X_i$ E X_j SONO INDIPENDENTI

→ TEOR. $X \sim N(m, K)$ È ASS. CONT. $\Leftrightarrow$ $\det K \neq 0$

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^d \, |\det K|}} \, e^{-\frac{1}{2}\langle x - m, K^{-1}(x-m)\rangle}$$

→ TEOR. $\forall m \in \mathbb{R}^d$, $\forall K \in \mathbb{R}^{d \times d}$ SIMMETRICA E SEMI-DEF. POS.,

$\exists \; X \sim N(m, K)$.

- K È SIMMETRICA E SEMI-DEF. POS. $\iff$

$\forall i,j \quad K_{ij} = K_{ji}$ $\qquad \langle a, Ka \rangle = \sum_{i,j} K_{ij} a_i a_j \geq 0 \quad \forall a \in \mathbb{R}^d$

$\Downarrow \qquad\qquad\qquad \Updownarrow$

$\exists\ \lambda_1, \dots, \lambda_d$ AUTOVALORI REALI T.C. $\lambda_i \geq 0 \quad \forall i = 1, \dots, d$

$\det K = \lambda_1 \cdots \lambda_d \neq 0 \iff \lambda_i > 0 \quad \forall i = 1, \dots, d$

- $X \sim N(0, I)$ CON I = IDENTITÀ $(I_{ij} = \delta_{ij})$
SI DICE NORMALE STANDARD IN $\mathbb{R}^d$.

- $\boxed{\text{ES.}}$ SIA K SIMM. E SEMI-DEF. POS.
SIA $A \in O(d)$ $[AA^* = A^*A = I]$ T.C. $A^*KA = D$
CON $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$.
DEF $\sqrt{D} := \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})$.
MOSTRARE CHE

$$Z \sim N(0, I) \implies X := m + A\sqrt{D}\,Z \sim N(m, K).$$

- CONVERGENZA

DEF. $(X_n)_{n \in \mathbb{N}}$, X V.A. IN E = SP. TOP. $[E = \mathbb{R}^d]$

$X_n \xrightarrow{d} X \iff E[\varphi(X_n)] \to E[\varphi(X)] \quad \forall \varphi: E \to \mathbb{R}$ CONT. E LIM.

$\mu_{X_n} \xrightarrow{d} \mu_X \iff \int \varphi \, d\mu_{X_n} \to \int \varphi \, d\mu_X$

$\boxed{\text{TEOR.}}$ SE $X_n \sim N(m_n, K_n)$ ALLORA $X_n \xrightarrow{d} X$

$\iff m_n \to m$, $K_n \to K$. IN TAL CASO $X \sim N(m, K)$.